Linear systems of curves

let $d \geq 1$, and $M_1, \dots, M_N$ the monomials in $k[x,y,z]$ of degree d. What is N?

$$\underbrace{\bullet\ \bullet\ \bullet\ \bullet}_{x}\ \Big|\ \underbrace{\bullet\ \bullet\ \bullet\ \bullet}_{y}\ \Big|\ \underbrace{\bullet\ \bullet\ \bullet}_{z} \qquad \text{d dots total, 2 bars}$$

so $$N = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$$

If F is a curve of deg d, then $F = \sum a_i M_i$, $a_i \in k$, not all 0.

$F \sim G \iff G = \sum \lambda a_i M_i$, some $\lambda \neq 0$.

i.e $[a_1 : \dots : a_N] \in \mathbb{P}^{N-1}$ determines a unique plane curve.

Thus, there's a bijection:

$$\left\{\begin{matrix}\text{plane curves} \\ \text{of deg } d\end{matrix}\right\} \longleftrightarrow \mathbb{P}^{N-1} = \mathbb{P}^{d(d+3)/2}$$

Ex: $\underset{\sim}{d=1}$: Each line $ax+by+cz$ corresponds to $[a:b:c] \in \mathbb{P}^2$. (Dual projective space — see HW).

$\underset{\sim}{d=2}$: The conic $a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$ corr. to the point $[a_1 : \dots : a_6] \in \mathbb{P}^5$, i.e. the set of plane conics is "parametrized" by a $\mathbb{P}^5$.

We can impose conditions on our curves of degree d and get a subset of $\mathbb{P}^{d(d+3)/2}$:

Def: If $V \subseteq \mathbb{P}^{\overbrace{d(d+3)/2}^{N-1}}$ is a linear subvariety, where the points of $\mathbb{P}^{N-1}$ correspond to plane curves of degree d, V is called a <u>linear system</u> of plane curves.

Ex: Let V = the set of lines through $[0:0:1]$, i.e. $ax+by+cz$ s.t. $c=0$. Then $V = \{[a:b:0]\} \subseteq \mathbb{P}^2$, a line in $\mathbb{P}^2$.

More generally...

Lemma: 1.) Fix a point $P \in \mathbb{P}^2$. The set of curves of degree d that contain P forms a hyperplane in $\mathbb{P}^{N-1}$.

2.) If $T: \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is a projective change of coordinates, Then $F \longmapsto T^*(F)$ is a projective change of coordinates of $\mathbb{P}^{N-1}$.

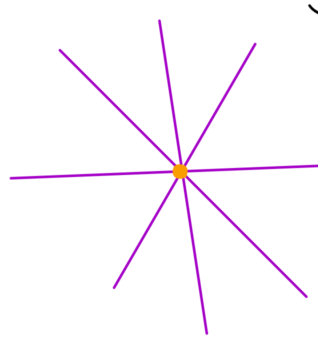
Pf: 1.) Let $M_1, \ldots, M_N$ be the monomials of degree d and $P = [a:b:c]$. Then $M_i(a,b,c) \neq 0$ for some i.

The curve corresponding to $[\alpha_1 : \ldots : \alpha_N]$ passes through P
$\iff \sum \alpha_i \underbrace{M_i(a,b,c)}_{\in k} = 0$

This is a linear form in the α_i, so it cuts out a hyperplane in $\mathbb{P}^{N-1}$.

2.) Similar idea (exer). $\square$

Ex: The linear system of lines through a point forms a line in the dual $\mathbb{P}^2$.

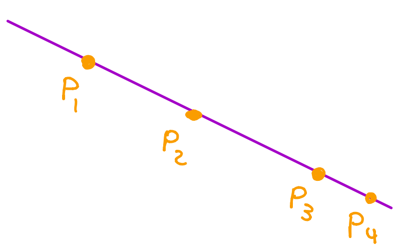


Question: Does passing through m points cut the dimension of the corresponding linear system by m?

Answer: It depends how the m hyperplanes intersect.

Ex: $V = \{\text{conics through } P_1, P_2, P_3, P_4\} \subseteq \mathbb{P}^5 = \{\text{all conics}\}$

Expect: $\dim V = 1$. If P_1, P_2, P_3, P_4 are collinear...



Let $F \in V$. Then $F \cap \ell > 2$, so F contains ℓ as a component.

Thus, F is the union of ℓ and any other line, so V is in one-to-one correspondence w/ set of lines $\Rightarrow \dim V = 2$.

Remark: let $N = \frac{d(d+3)}{2}$. The intersection of N hyperplanes in $\mathbb{P}^N$ is nonempty, so there is some curve of degree d passing through any given N points.

What if we require the curves of degree d to have high multiplicity at some fixed point?

Ex: $P = [0:0:1] \in \mathbb{P}^2$. If F is a conic through P, then

$$F = a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz \quad (\text{no } z^2 \text{ term})$$

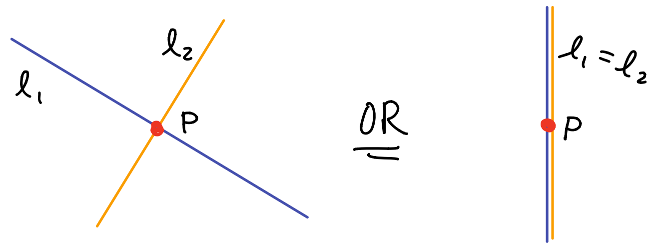
What if $V = \{F \mid F \text{ is a conic and } m_P(F) \geq 2\}$?

Since $P \in U_3$, we can dehomogenize with respect to z and get

$$f = a_1 x^2 + a_2 xy + a_3 x + a_4 y^2 + a_5 y, \text{ so } a_3 = a_5 = 0$$

$\Rightarrow$ $F = ax^2 + bxy + cy^2$, which factors into two linear factors.

$\Rightarrow$ $\dim V = 2$, and the set of conics not simple at P is exactly pairs of lines through P:



More generally, let $P = [0:0:1]$ and $r \leq d+1$. let

$V = \{ F \mid \deg F = d \text{ and } m_p(F) \geq r \}$.

If $F = \sum \underset{\substack{\uparrow \\ \text{in } k}}{a_i} \underset{\nwarrow \text{monomial}}{M_i} \in V$, then $a_i = 0$ if $M_i = x^a y^b z^c$ and $a+b<r$.

There are $\frac{r(r+1)}{2}$ such coefficients, so $\dim V = \frac{d(d+3)}{2} - \frac{r(r+1)}{2}$.

Notation: $P_1, \ldots, P_n \in \mathbb{P}^2$ distinct points, $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$.

$$V_d(r_1 P_1, \ldots, r_n P_n) := \{ \text{curves of degree } d \mid m_{P_i}(F) \geq r_i \}$$

Theorem: 1.) $V = V_d(r_1 P_1, \ldots, r_n P_n)$ is a linear subvariety of $\mathbb{P}^N$ of dimension $\geq \underbrace{\frac{d(d+3)}{2}}_{N} - \sum \frac{r_i(r_i+1)}{2}$

2.) If $d \geq \left(\sum r_i\right) - 1$, then the equality in part 1.) holds.

Pf: 1.) Same argument as above.

2.) Let $m = \left(\sum r_i\right) - 1$.

If $m = 1$, then $r_1 = 2$, the rest 0, so we're done by above.

<u>OR</u> $r_1 = 1$, $r_2 = 1$, the rest 0. Then $V =$ intersection of 2 distinct hyperplanes, which always has codim 2 (2 distinct forms will always be lin. indep), so we're done.

Thus, we can assume $m, d > 1$.

We'll prove by induction on m:

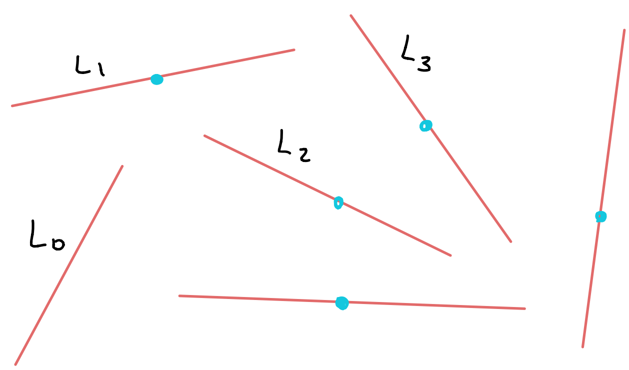
Case 1: Each $r_i = 1$. Let $V_i = V_d(P_1, \dots, P_i)$.

Note that $V_1 \supseteq V_2 \supseteq \dots \supseteq V_{n-1} \supseteq V_n$.

By induction, $\dim V_{n-1} = N - \sum_{i=1}^{n-1} \frac{r_i(r_i+1)}{2} = N-(n-1)$, and by 1.), $\dim V_n \geq N - n$.

So we just need to show $V_n \neq V_{n-1}$.

For each i, let L_i be a line through P_i but not through $P_j \neq P_i$. Let L_0 be a line not through any P_i.



Then $F = L_1 \cdots L_{n-1} L_0^{d-(n-1)} \in V_{n-1}$ but $F \notin V_n$, so $V_n \neq V_{n-1}$.

$$d-(n-1) = d - (\Sigma r_i - 1) \geq 0$$

Case 2: Some $r_i > 1$. WLOG, assume $r_1 > 1$ and $P_1 = [0:0:1]$.

Let $V_0 = V_d((r_1 - 1)P_1, r_2 P_2, \dots, r_n P)$.

For $F \in V_0$, its dehomog. w/ respect to z is

$$f = \sum_{i=0}^{r_1 - 1} a_i x^i y^{r-1-i} + \text{higher terms.}$$

Let $V_i = \{ F \in V_0 \mid a_j = 0 \text{ for } j < i \}$.

Then $V_0 \supset V_1 \supset \dots \supset V_{r_1} = V_d(r_1 P_1, \dots, r_n P_n)$.

Note that $\frac{(r_1 - 1)(r_1 - 1 + 1)}{2} = \frac{r_1(r_1+1)}{2} - r_1$.

So we need to show $\dim V_{r_1}$ is exactly r_1 less than $\dim V_0$, so it suffices to show $V_i \neq V_{i+1}$.

Let $W_0 = V_{d-1}((r_1 - 2)P_1, r_2 P_2, \dots, r_n P_n)$.

For $F \in W_0$, its dehomogenization looks like

$$f = \sum a_i x^i y^{r-2-i} + \text{higher terms.}$$

Let $W_i = \{ F \in W_0 \mid a_j = 0 \text{ for } j < i \}$

By induction, $W_0 \supsetneq W_1 \supsetneq \dots \supsetneq W_{r_1 - 1} = V_{d-1}((r_1 - 1)P_1, r_2 P_2, \dots, r_n P_n)$ since

$$\dim W_0 - \dim W_{r_1 - 1} = \frac{(r_1 - 1)(r_1)}{2} - \frac{(r_1 - 2)(r_1 - 1)}{2} = \frac{r_1 - 1}{2} \cdot 2 = r_1 - 1$$

and each successive dimension differs by at most one.

For $i = 0, 1, \ldots, r_1 - 2$, choose $F_i \in W_i \setminus W_{i+1}$.

Then $yF_i \in V_i \setminus V_{i+1}$.

Also, $F_{r_1-2} \in W_{r_1-2} \setminus W_{r_1-1}$, so $F_{r_1-2} = \underset{\neq 0}{a_{r-2}}\, x^{r-2}$ + higher terms

So $xF_{r-2} = a_{r-2}x^{r-1}$ + higher terms $\in V_{r_1-1}$ but not in V_{r_1}.

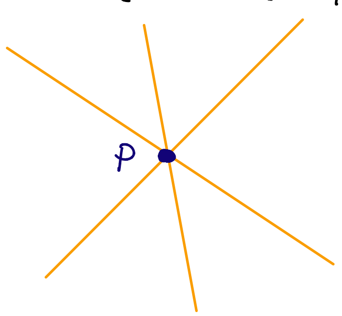
Thus, $V_i \neq V_{i+1}$ for $i = 0, \ldots, r-1$. $\square$

Ex: Let $V = V_3(3P)$ where $P = [0:0:1]$.

By the theorem, since $3 \geq 3-1$, $\dim V = \frac{3(3+3)}{2} - \frac{3(3+1)}{2} = 9 - 6 = 3$.

Let $F \in V$ and $Q \neq P$ another point on F. Then L_{PQ} intersects F in multiplicity $\geq 4 \implies L_{PQ}$ is a component of F.

$\implies V$ consists of the cubics w/ three linear factors through P.



So there is a $\mathbb{P}^3$ worth of triples of lines through a point in $\mathbb{P}^2$.